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Supplement to The Numerical Solution of Second-Order Boundary Value Problems on Nonuniform Meshes

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APPENDIX A

LEMMA A.1. Let the leading terms in the truncation error be

$$(T)_{i-1/2}=\frac{\Delta_{i+1}-2\Delta_i+\Delta_{i-1}}{\Delta_i}\,p(x_{i-1/2}) \tag{A.1}$$
$$+\frac{F(\Delta_{i+1},\Delta_i)-F(\Delta_i,\Delta_{i-1})}{\Delta_i}\,q(x_{i-1/2})+O(\Delta^2)\ ,\quad i=2,\ldots,N-1\ ,$$

where $p(x)\varepsilon C^2(0,1)$ and $q(x)\varepsilon C^1(0,1)$ and

$$\max_{1\le i\le N-1}|F(\Delta_{i+1},\Delta_i)|\le C\ \Delta^2_{\max}$$

for some class of meshes M. Then we can write

$$T=D_0D_1T_1+D_0T_2+T_3\ , \tag{A.2}$$

where $\|T_i\|\le C\ \Delta^2_{\max}, i=1,2,3$, for all meshes in M.

Proof. First, we will break the proof into two parts: let

$$T=\Sigma_1+\Sigma_2\ ,$$

where

$$(\Sigma_1)_{i-1/2}=\frac{\Delta_{i+1}-2\Delta_i+\Delta_{i-1}}{\Delta_i}\,p(x_{i-1/2})$$

and

$$(\Sigma_2)_{i-1/2}=\frac{F(\Delta_{i+1},\Delta_i)-F(\Delta_i,\Delta_{i-1})}{\Delta_i}\,q(x_{i-1/2})\ .$$

Using Taylor series expansion on $p_{i-1/2}$ gives us

$$(\Sigma_1)_{i-1/2}=\frac{\Delta_{i+1}-\Delta_i}{\Delta_i}\,p_i-\frac{\Delta_i-\Delta_{i-1}}{\Delta_i}\,p_{i-1}$$
$$-\frac{1}{2}(\Delta_{i+1}-\Delta_i)\,p_i'-\frac{1}{2}(\Delta_i-\Delta_{i-1})\,p_{i-1}' \tag{A.3}$$
$$+\frac{1}{8}(\Delta_{i+1}-\Delta_i)\Delta_i p''-\frac{1}{8}(\Delta_i-\Delta_{i-1})\Delta_i p''\ ,$$

where throughout this proof we will denote intermediate values by leaving the function p or q without subscripts. Again using Taylor series, for the first line of (A.3) we have

$$\frac{\Delta_{i+1}-\Delta_i}{\Delta_i}\,p_i-\frac{\Delta_i-\Delta_{i-1}}{\Delta_i}\,p_{i-1}=\left[-\frac{1}{\Delta_i}\ \frac{1}{\Delta_i}\right]\begin{bmatrix}\dfrac{\Delta_i^2\,p_{i-1/2}-\Delta_{i-1}^2p_{i-3/2}}{\Delta_i+\Delta_{i-1}}\\[2ex]\dfrac{\Delta_{i+1}^2\,p_{i+1/2}-\Delta_i^2\,p_{i-1/2}}{\Delta_{i+1}+\Delta_i}\end{bmatrix}$$

$$+\frac{1}{2}\left[-\frac{1}{\Delta_i}\ \frac{1}{\Delta_i}\right]\begin{bmatrix}-\dfrac{\Delta_i^3}{\Delta_i+\Delta_{i-1}}\,p'-\dfrac{\Delta_{i-1}^3}{\Delta_i+\Delta_{i-1}}\,p'\\[2ex]-\dfrac{\Delta_{i+1}^3}{\Delta_{i+1}+\Delta_i}\,p'-\dfrac{\Delta_i^3}{\Delta_{i+1}+\Delta_i}\,p'\end{bmatrix} \tag{A.4}$$

$$= \frac{1}{2}\left[D_0 D_1 \begin{bmatrix} \cdot \\ \cdot \\ \Delta_i^2 p_{i-1/2} \\ \cdot \\ \cdot \end{bmatrix} + D_0 \left[-\frac{\Delta_i^3}{\Delta_i + \Delta_{i-1}} p' - \frac{\Delta_{i-1}^3}{\Delta_i + \Delta_{i-1}} p' \right] \right]_{i-1/2}$$

The next line in (A.3) is even easier. We note that

$$-\frac{1}{2}(\Delta_{i+1} - \Delta_i) p_i' - \frac{1}{2}(\Delta_i - \Delta_{i-1}) p_{i-1}' = -\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\Delta_i^2 - \Delta_{i-1}^2}{\Delta_i + \Delta_{i-1}} p_{i-1}' \\ \frac{\Delta_{i+1}^2 - \Delta_i^2}{\Delta_{i+1} + \Delta_i} p_i' \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\Delta_i^2 p_{i-1/2}' - \Delta_{i-1}^2 p_{i-3/2}'}{\Delta_i + \Delta_{i-1}} \\ \frac{\Delta_{i+1}^2 p_{i+1/2}' - \Delta_i^2 p_{i-1/2}'}{\Delta_{i+1} + \Delta_i} \end{bmatrix} + O(\Delta^2)$$

$$= \left[-D_0 \bar{A}_1 \begin{bmatrix} \cdot \\ \cdot \\ \frac{1}{2}\Delta_i^2 p'_{i-1/2} \\ \cdot \\ \cdot \end{bmatrix} \right]_{i-1/2} + O(\Delta^2) \; .$$

The last term in (A.3) is $O(\Delta^2)$ as it stands. Clearly, then, we can write Σ_1 in the form of (A.2).

For the Σ_2 term, we also employ Taylor series to get

$$(\Sigma_2)_{i-1/2} = \frac{F(\Delta_{i+1}, \Delta_i)}{\Delta_i} q_i - \frac{F(\Delta_i, \Delta_{i-1})}{\Delta_i} q_{i-1}$$

$$- F(\Delta_{i+1}, \Delta_i) q' - F(\Delta_i, \Delta_{i-1}) q'$$

$$= \left[D_0 \begin{bmatrix} \cdot \\ \cdot \\ F(\Delta_i, \Delta_{i-1}) q_{i-1} \\ \cdot \\ \cdot \end{bmatrix} \right]_{i-1/2} + O(\Delta^2) \; .$$

Thus, we can also write Σ_2 in the form of (A.2). ●

In the proof of the main theorem of section 5, we will assume that the truncation error is given in the form of (A.2). Lemma A.1 provides sufficient conditions for this to be true.

Recalling (A.4), we see again that the $O(1)$ term in the truncation error can be written as $D_0 D_1 T_1$. Throughout the remainder of this section, we will define

(A.5) $$(T_1)_{i-1/2} \equiv \frac{1}{2} \Delta_i^2 \, p(x_{i-1/2}) \; ,$$

where $p(x)$ is smooth. Lemma A.2 deals with writing the truncation error term $L_1 T_1$ as $D_0 \bar{T}_1$, where $\bar{T}_1 = O(\Delta^2)$.

LEMMA A.2. Let T_1 be the truncation error as given in (A.5). Let the matrix L_1 represent the three-point difference approximation to $a(x)y' + b(x)y$, namely

$$L_1 = \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 & & \\ & \alpha_3 & \beta_3 & \gamma_3 & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{bmatrix}_{(N-2)\times N}$$

Further, let $|\Delta_{i+1}\alpha_{i+1} + \Delta_i \beta_i + \Delta_{i-1}\gamma_{i-1}| \le C\,\Delta_{\max}$, $i = 2, \ldots, N-1$ and $|\Delta_i \alpha_i|, |\Delta_i \gamma_i| \le C$ for any mesh in some class M. Then,

(A.6a) $$L_1 T_1 = D_0 \bar{T}_1 + \bar{T}_6 \; ,$$

where

(A.6b) $$\|\bar{T}_1\|_\infty \le C\,\Delta_{\max}^2 \; ,$$

(A.6c) $$\|\bar{T}_6\|_\infty \le C\,\Delta_{\max}^2 \; ,$$

for any mesh in M.

Proof. We can write L_1 in the following way:

$$L_1 = D_0 L_2 + L_3 \; ,$$

where

$$L_2 = \begin{bmatrix} -\Delta_2\alpha_2 & \Delta_1\gamma_1 & & & \\ & -\Delta_3\alpha_3 & \Delta_2\gamma_2 & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & -\Delta_N\alpha_N & \Delta_{N-1}\gamma_{N-1} \end{bmatrix}_{(N-1)\times N} ,$$

$$L_3 = \begin{bmatrix} 0 & \frac{1}{\Delta_2}[\Delta_3\alpha_3 + \Delta_2\beta_2 + \Delta_1\gamma_1] & 0 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \quad \cdot \\ & & 0 & \frac{1}{\Delta_{N-1}}[\Delta_N\alpha_N + \Delta_{N-1}\beta_{N-1} + \Delta_{N-2}\gamma_{N-2}] & 0 \end{bmatrix}_{(N-2)\times N} ,$$

where we note that γ_1 and α_N are arbitrary. Now, by hypothesis, $|\Delta_i \alpha_i|$ and $|\Delta_i \gamma_i| \le C$, so that

$$\|L_2\|_\infty \le C ,$$

and thus,

$$\|\bar{T}_1\|_\infty = \|L_2 T_1\|_\infty \le \|L_2\|_\infty \|T_1\|_\infty \le C\Delta_{\max}^2$$

Also,

$$(L_3 T_1)_i = \frac{1}{\Delta_i}(\Delta_{i+1}\alpha_{i+1} + \Delta_i\beta_i + \Delta_{i-1}\gamma_{i-1})\frac{1}{2}\Delta_i^2 p(x_{i-1/2})$$

for $i = 2, \ldots N-1$. Thus,

$$|(L_3 T_1)_i| \le |\Delta_{i+1}\alpha_{i+1} + \Delta_i\beta_i + \Delta_{i-1}\gamma_{i-1}|\Delta_{\max} \max_{[0,1]}|p| ,$$

and

$$\|L_3 T_1\|_\infty = \|\bar{T}_4\|_\infty \le C\Delta_{\max}^2 \quad \bullet$$

At this point, it is instructive to look at a difference approximation of $ay' + by$ and examine the application of this Lemma. Suppose we examine the scheme

(A.7)
$$a_{i-1/2}\left[\frac{1}{2}\left[\frac{v_{i+1/2} - v_{i-1/2}}{\Delta_{i+1/2}}\right] + \frac{1}{2}\left[\frac{v_{i-1/2} - v_{i-3/2}}{\Delta_{i-1/2}}\right]\right] + b_{i-1/2}v_{i-1/2} .$$

The elements of the matrix L_1 are

$$[\alpha_i, \beta_i, \gamma_i] = \left[-\frac{a_{i-1/2}}{2\Delta_{i-1/2}} , \frac{a_{i-1/2}}{2}\left[\frac{1}{\Delta_{i-1/2}} - \frac{1}{\Delta_{i+1/2}}\right] + b_{i-1/2} , \frac{a_{i-1/2}}{2\Delta_{i+1/2}}\right]$$

We have to check two properties in order for this Lemma to apply. The easiest is boundedness for $\Delta_i\alpha_i$, $\Delta_i\beta_i$, and $\Delta_i\gamma_i$. If a and b are bounded, then clearly for any mesh,

$$|\Delta_i\alpha_i|, |\Delta_i\beta_i|, |\Delta_i\gamma_i| \le \max_{(0,1)}(|a| + |b|) .$$

The other quantity of interest is

$$\Delta_{i+1}\alpha_{i+1} + \Delta_i\beta_i + \Delta_{i-1}\gamma_{i-1}$$

$$= a_{i-1/2}\left[-\frac{\Delta_{i+1}}{\Delta_{i+1} + \Delta_i} + \frac{\Delta_i}{\Delta_i + \Delta_{i-1}} - \frac{\Delta_i}{\Delta_{i+1} + \Delta_i} + \frac{\Delta_{i-1}}{\Delta_i + \Delta_{i-1}}\right] + \Delta_i b_{i-1/2}$$

$$- \frac{\Delta_{i+1}}{\Delta_{i+1} + \Delta_i}(a_{i+1/2} - a_{i-1/2}) + \frac{\Delta_{i-1}}{\Delta_i + \Delta_{i-1}}(a_{i-3/2} - a_{i-1/2})$$

$$= \Delta_i b_{i-1/2} - \frac{\Delta_{i+1}}{2}\left[\frac{a_{i+1/2} - a_{i-1/2}}{\Delta_{i+1/2}}\right] - \frac{\Delta_{i-1}}{2}\left[\frac{a_{i-1/2} - a_{i-3/2}}{\Delta_{i-1/2}}\right]$$

Now, if b is continuous and $a \in C^1(0,1)$, then

$$|\Delta_{i+1}\alpha_{i+1} + \Delta_i\beta_i + \Delta_{i-1}\gamma_{i-1}| \le \max_{(0,1)}(|a'| + |b|)\Delta_{\max} .$$

It is easy to construct examples of consistent approximations to $ay' + by$ that do not satisfy the hypothesis of this Lemma. Numerical experiments indicate such schemes may not be truly second-order.

Lemma A.3 is the exact counterpart for cell-centered meshes of Lemma 2.2.

LEMMA A.3. On any cell-centered mesh, define an N-vector, E_2, by

(A.8a)
$$(E_2)_{1/2} = 0$$

(A.8b)
$$(E_2)_{i-1/2} = \sum_{j=1}^{i-1}\left[\frac{\Delta_{j+1} + \Delta_j}{2}\right](T_2)_j , \quad i = 2, \ldots, N .$$

Then,

(A.9)
$$D_0 D_1 E_2 = D_0 T_2 ,$$

and, furthermore,

(A.10a)
$$\|E_2\|_\infty \le |x_{N-1/2} - x_{1/2}|\|T_2\|_\infty ,$$

(A.10b)
$$\|D_1 E_2\|_\infty = \|T_2\|_\infty$$

Proof. The proof is similar to that of Lemma 2 2 ●

Aside from considerations involving the boundary conditions, the only remaining hurdle is to show that L_1E_2 is $O(\Delta_2)$. In Section 3, the equivalent work is contained in the Corollary to Theorem 3.2.

LEMMA A.4. Let L_1 represent the three-point difference approximation to $a(x)y' + b(x)y$ in our difference scheme, and let $E_2 T_2$ be as discussed in Lemma A.3. Further, let L_1 be a consistent approximation to $ay' + by$ on some class of meshes M such that

$$\max_i(|\gamma_i\Delta_{i+1/2}|, |\alpha_i\Delta_{i-1/2}|) \le C$$

Then,

$$\|L_1E_2\| \le C(\|E_2\|_\infty + \|T_2\|_\infty)$$

Proof We have, by definition,

$$(L_1E_2)_i = \alpha_i(E_2)_{i-3/2} + \beta_i(E_2)_{i-1/2} + \gamma_i(E_2)_{i+1/2}$$

$$= (E_2)_{i-1/2}(\alpha_i + \beta_i + \gamma_i) + (\gamma_i\Delta_{i+1/2}(T_2)_i - \alpha_i\Delta_{i-1/2}(T_2)_{i-1})$$

Consistency requires that $|\alpha_i + \beta_i + \gamma_i| \le C$. Furthermore, we have

$$|\gamma_i\Delta_{i+1/2}|, |\alpha_i\Delta_{i-1/2}| \le C ,$$

so that

$$|(L_1E_2)_i| \le C\,|(E_2)_{i-1/2}| + C(|T_2|_i + |T_2|_{i-1}) ,$$

and thus

$$\|L_1E_2\|_\infty \le C(\|E_2\|_\infty + \|T_2\|_\infty) . \quad \bullet$$

For our sample difference scheme (5.20), we have

$$|\alpha_i + \beta_i + \gamma_i| = |b_{i-1/2}| \le \max_{(0,1)}|b(x)| ,$$

and

$$\Delta_{i+1/2}\gamma_i = \frac{1}{2}a_{i-1/2} , \quad \Delta_{i-1/2}\alpha_i = -\frac{1}{2}a_{i-1/2} ,$$

so that

$$|\Delta_{i+1/2}\gamma_i|, |\Delta_{i-1/2}\alpha_i| \le \frac{1}{2}\max_{(0,1)}|a(x)| .$$

Thus, this difference scheme easily satisfies the hypothesis of Lemma A.4